Tutorial Notes 8

1. Suppose that $F = \nabla f$ is a conservative vector field and
$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} F \cdot \mathrm{d}r.$$
Prove that $\nabla g = F$.

   Solutions:

   Since $g(x, y, z) = f(x, y, z) - f(0, 0, 0)$, $\nabla g = \nabla f = F$.

2. (a) Find a potential function for the gravitational field
$$F = -GmM \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

   (b) Let P_1 and P_2 are two points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in moving a particle form P_1 to P_2 is $GmM(s_2^{-1} - s_1^{-1})$.

   Solutions:

   (a) For (x, y, z), calculate
$$-\int_\gamma F \cdot \mathrm{d}r$$
where $\gamma(t) = (xt, yt, zt)$, $1 \leq t < \infty$, which is the candidate of the potential function. Then it is equal to
$$\int_1^\infty \frac{GmM}{(x^2 + y^2 + z^2)^{1/2}} \cdot \frac{1}{t^2} \,\mathrm{d}t = \frac{GmM}{(x^2 + y^2 + z^2)^{1/2}}.$$
It is easy to see that $GmM(x^2 + y^2 + z^2)^{-1/2}$ is a potential function.

   (b) The work is the difference of the potential function, which is $GmM(s_2^{-1} - s_1^{-1})$.

3. Find the outward flux of the vector field
$$\left(3xy - \frac{x}{1 + y^2}, \mathrm{e}^x + \arctan y \right)$$
across the cardioid $r = a(1 + \cos\theta)$.

   Solutions:

The divergence is

$$3y - \frac{1}{1+y^2} + \frac{1}{1+y^2} = 3y.$$

By symmetry, the flux is 0.

4. (a) Prove Green's area formula

$$|\Omega| = \frac{1}{2} \int_{\partial\Omega} (x\,\mathrm{d}y - y\,\mathrm{d}x).$$

(b) Calculate the area of the region enclosed by the ellipse $\gamma(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$.

Solutions:

(a) The curl of the vector field is 2. Hence Green's area formula holds.

(b) It suffices to calculate

$$\frac{1}{2} \int_{\gamma} (x\,\mathrm{d}y - y\,\mathrm{d}x),$$

which is equal to

$$\frac{1}{2} \int_0^{2\pi} [a\cos t(b\cos t) - b\sin t(-a\sin t)]\,\mathrm{d}t = \pi ab.$$

5. C is a boundary of a region where Green's theorem holds. Calculate

(a)

$$\int_C (f(x)\,\mathrm{d}x + g(y)\,\mathrm{d}y);$$

(b)

$$\int_C (ky\,\mathrm{d}x + hx\,\mathrm{d}y).$$

Solutions:

(a) Since the curl is

$$[g(y)]_x - [f(x)]_y = 0,$$

the integral is 0.

(b) Since the curl is

$$(hx)_x - (ky)_y = h - k,$$

the integral is $(h-k)|\Omega|$ where C is the boundary of Ω.